ON THE RECURSIVE SEQUENCE

$$x_{n+1} = \alpha + \frac{x_{n-k}}{f(x_n, ..., x_{n-k+1})}$$

G. L. KARAKOSTAS AND S. STEVIĆ

Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

Matematički Fakultet, Studentski Trg 16, 11000 Beograd, Serbia

e-mail: gkarako@cc.uoi.gr, sstevic@ptt.yu; sstevo@matf.bg.ac.yu

Abstract

The boundedness, global attractivity, oscillatory and periodicity of the nonnegative solutions of the difference equation of the form

$$x_{n+1} = \alpha + \frac{x_{n-k}}{f(x_n, ..., x_{n-k+1})}, \qquad n = 0, 1, ...$$

is investigated, where $\alpha \geq 0$, $k \in \mathbf{N}$ and $f : \mathbf{R}_+^k \rightarrow \mathbf{R}_+$ is a continuous function nondecreasing in each variable.

1 Introduction

We investigate the behavior of the (positive) solutions of a difference equation of the form

$$x_{n+1} = \alpha + \frac{x_{n-k}}{f(x_n, ..., x_{n-k+1})}, \qquad n = 0, 1, ... \tag{1}$$

for the various values of the nonnegative parameter α. The motivation of this paper is the second order nonlinear rational recursive sequence

$$x_{n+1} = \beta + \frac{x_{n-1}}{x_n}, \quad n = 1, 2, ... \tag{2}$$

where β as well as the initial values x_0, x_1 are positive real numbers, posed in an open problem by Ladas in [11], where the following conjecture was formulated, which almost at the same time also was discussed in [1]:

2000 Mathematics Subject Classification: Primary 39A10

Key words and phrases: Equilibrium, positive solution, difference equation, boundedness, global attractivity, periodic nature

Conjecture 5.2.4 [11]: *Every positive solution of Equ.* (2) *is bounded if and only if* $\beta \geq 1$. *Furthermore, if* $\beta = 1$, *then every positive solution converges to a period two solution and, if* $\beta > 1$, *then every positive solution converges to the equilibrium* $\beta + 1$.

A more general version of (2) was discussed in [4], where the behavior of the positive solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + x_n}, \quad n = 0, 1, ... \tag{3}$$

is investigated. Again here the parameters α, β, γ, A and the initial conditions x_{-1} and x_0 are positive real numbers. By the change of variables $x_n = Ay_n$ Equ. (3) reduces to the difference equation

$$y_{n+1} = \frac{p + qy_n + ry_{n-1}}{1 + y_n}, \quad n = 0, 1, ... \tag{4}$$

where

$$p = \frac{\alpha}{A^2} \quad q = \frac{\beta}{A}, \quad \text{and} \quad r = \frac{\gamma}{A}.$$

The following trichotomy result was proved in [4]:

Theorem A. *Consider Equ.* (4). *If it holds* $r = q + 1$, *then every solution converges to period-two solution, if* $r < q + 1$, *then the equilibrium is globally asymptotically stable and if* $r > q + 1$, *then there exist unbounded solutions.*

The statement in case $r = q + 1$ is a special case of the main result in [17]. One of the first results of this kind is Theorem 5.1 in [3]. In the articles [2], [13] and [14], some global convergence results were obtained which can be applied to some nonlinear difference equations in proving that every solution of these ones converge to a period-two solution, see also [15]. In [7] the full limiting sequences method is used to prove a result of such kind. A closely related result concerning the periodic character also can be found in [20].

Assuming for a moment that $p = q =: \beta$ Equ. (4) becomes

$$y_{n+1} = \beta + \frac{y_{n-1}}{\frac{1}{r}(1 + y_n)}, \quad n = 0, 1, ... \tag{5}$$

Hence if we set $g(u) := \frac{1}{r}(1+u)$, then Theorem A says that if it holds $g(\beta) = 1$, then every solution of Equ. (5) converges to a period-two solution, if $g(\beta) > 1$, then the equilibrium is globally asymptotically stable and if $g(\beta) < 1$, then there exist unbounded solutions. Also, if $g(u) = u$, Conjecture 5.2.5 in [11] and

Theorem 4.1 in [1] say that if $g(\beta) = 1$, then every solution of Equ. (2) converges to a period-two solution, while Theorem 5.2 in [1] says that if $g(\beta) > 1$, then every positive solution converges to the equilibrium.

Motivated by these observations in this paper we investigate the boundedness character, the global attractivity as well as the oscillatory and periodic nature of the nonnegative solutions of Equ. (1) where the parameter α is a nonnegative real number, k is a fixed positive integer and f is a function satisfying the following condition:

(H) $f : \mathbf{R}_+^k \to \mathbf{R}_+$ *is a given continuous function, nondecreasing in each variable and increasing in at least one.*

In what follows we shall assume that the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are positive real numbers. Then under the condition (H) the corresponding solution (x_n) has positive terms.

The above mentioned properties which we seek for Equ. (1) were also discussed (see, e.g. [2, 3, 6-20]) for some nonlinear difference equations of several forms.

In our analysis an important role plays the function

$$g(u) := f(u, ..., u), \quad u > 0.$$

It is clear that under condition (H) the function g is increasing.

2 Semicycle analysis about a positive equilibrium

A *positive semicycle* of a solution (x_n) about a positive equilibrium $\bar{x}$ consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all greater than or equal to $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} < \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \bar{x}.$$

A *negative semicycle* of a solution (x_n) about $\bar{x}$ consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all less than to $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \bar{x}.$$

The first semicycle of a solution starts with the term x_{-1} and is positive if $x_{-1} \geq \bar{x}$ and negative if $x_{-1} < \bar{x}$.

We say that a sequence (x_n) oscillates about x^* if for every $n_0 \in \mathbf{N}$ there are $p, q \geq n_0$ such that $(x_p - x^*)(x_q - x^*) \leq 0$.

The following theorem is the main result in this section and it generalizes Theorem 3.2 in [3].

Theorem 1. *Let $k \in \mathbf{N}$ be fixed and consider a continuous function H mapping the set $(0, \infty)^{k+1}$ into $(0, \infty)$ and having the following properties: there is an index $i_0 \in \{1, 2, ..., , k\}$ such that $H(z_1, ..., z_k, y)$ is nonincreasing in each $z_i, i \in \{1, ..., k\} \setminus \{i_0\}$, decreasing in z_{i_0}, and increasing in y. Let $\bar{x}$ be a positive equilibrium of the difference equation*

$$x_{n+1} = H(x_n, ..., x_{n-k+1}, x_{n-k}), \quad n = 0, 1, 2, ... \tag{6}$$

Then, except possibly for the first semicycle, every oscillatory solution with positive initial values has semicycles of length at most k.

Proof. Let (x_n) be an oscillatory solution of Equ. (6) with at least two semicycles. If a semicycle has length greater than or equal to k, then there is an $N \geq 0$ such that either

$$x_{N-k} < \bar{x} \leq x_{N-k+1}, ..., x_N \qquad \text{or} \qquad x_{N-k} \geq \bar{x} > x_{N-k+1}, ..., x_N.$$

Using the conditions of the theorem in the first case we obtain

$$x_{N+1} = H(x_N, ..., x_{N-k+1}, x_{N-k}) < H(\bar{x}, ..., \bar{x}) = \bar{x},$$

and in the second case we get

$$x_{N+1} = H(x_N, ..., x_{N-k+1}, x_{N-k}) > H(\bar{x}, ..., \bar{x}) = \bar{x},$$

as desired.

Corollary 1. *Assume that f is a given function satisfying assumption (H) and let K be a positive equilibrium of Equ. (1). Then, except possibly for the first semicycle, every oscillatory solution with positive initial values has semicycles of length at most k.*

3 The case $g(\alpha) > 1$

We start our analysis with the case $g(\alpha) > 1$. Since any solution of Equ. (1) with positive initial values stays above α, it follows that under the condition (H) it holds $g(u) > g(\alpha) > 1$, for all $u > \alpha$.

If $\alpha = 0$, then zero is the only nonnegative equilibrium. From Equ. (1) we get

$$x_{n+1} < \frac{x_{n-k}}{g(0)}, \quad n \in \mathbf{N}.$$

Thus the zero equilibrium is a geometrically global attractor for the (positive) solutions (see, Definition 1 in [12]).

Next assume that $\alpha > 0$. In the following theorem we give exact bounds for the solutions.

Theorem 2. *Assume that $g(\alpha) > 1$ and f satisfies assumption (H). Then Equ. (1) has a unique positive equilibrium K and every solution with positive initial conditions $x_{-k}, ..., x_0$ is bounded by the number*

$$M_0 := \max\{x_{-k}, ..., x_0\} + \frac{\alpha g(\alpha)}{g(\alpha) - 1}.$$

Proof. The equilibrium points of Equ. (1) satisfy the equation

$$F(x) := x - \alpha - \frac{x}{g(x)} = 0.$$

It is clear that the continuous function $F : [0, \infty) \to \mathbf{R}$ satisfies $F(\alpha) = -\frac{\alpha}{g(\alpha)} < 0$ and $\lim_{x \to +\infty} F(x) = +\infty$. Thus there is an $K \in (0, \infty)$ such that $F(K) = 0$. Now observe that for all $x > y \geq \alpha$ it holds

$$F(x) - F(y) = \frac{(x - y)g(x)(g(y) - 1) + x(g(x) - g(y))}{g(x)g(y)} > 0,$$

which means that F is increasing. Consequently K is the unique positive equilibrium point of Equ. (1).

Now we recall that $x_n > \alpha$, for $n \geq 1$. Thus we get

$$x_{n+1} = \alpha + \frac{x_{n-k}}{f(x_n, ..., x_{n-k+1})} < \alpha + \frac{x_{n-k}}{g(\alpha)}, \quad n = 0, 1, 2, ... \tag{7}$$

Let $q := 1/g(\alpha)$. From (7), by induction, we obtain

$$x_{(k+1)m+r+1} < x_{r-k}q^{m+1} + \alpha \sum_{j=0}^{m} q^j < x_{r-k} + \frac{\alpha}{1-q},$$

for all $m \in \mathbf{N} \cup \{0\}$ and $r \in \{0, 1, ..., k\}$. This inequality implies that $x_n \leq M_0$, for all n, as desired.

The following results refers to the global attractivity of positive solutions.

Theorem 3. *Assume that $\alpha > 0$, $g(\alpha) > 1$ and f satisfies assumption (H). If the function $x \longrightarrow (g(x) - g(\alpha))/(x - \alpha)$ is decreasing on $(\alpha, +\infty)$, then every positive solution of Equ. (1) converges.*

Proof. By Theorem 2 every positive solution of Equ. (1) is bounded. Hence there are finite $\liminf_{n\to\infty} x_n =: l \geq \alpha$ and $\limsup_{n\to\infty} x_n =: L \geq l$. It is clear that $g(L)$ is a finite real number. Letting lim inf and lim sup in (1) and by using the monotonicity of the function f we obtain

$$l \geq \alpha + \frac{l}{g(L)} \quad \text{and} \quad L \leq \alpha + \frac{L}{g(l)}, \tag{8}$$

which imply that

$$(l - \alpha)(g(L) - g(\alpha)) \geq \alpha g(\alpha) + (1 - g(\alpha))l$$

and

$$(L - \alpha)(g(l) - g(\alpha)) \leq \alpha g(\alpha) + (1 - g(\alpha))L.$$

If $l = \alpha$, then from the first inequality in (8) we get $l = 0 = \alpha$, a contradiction. Hence $l > \alpha$ and so $g(l) > g(\alpha)$.

Assume that $l < L$. Then from our assumption we get $(g(l) - g(\alpha))/(l - \alpha) > (g(L) - g(\alpha))/(L - \alpha)$ and therefore from the inequalities above it follows that

$$(1 - g(\alpha))l < (1 - g(\alpha))L,$$

which is a contradiction. Hence $l = L$.

Corollary 2. *Assume that $\alpha > 0$, $g(\alpha) > 1$ and f satisfies assumption (H). If the function g is strictly concave on $(\alpha, +\infty)$, then every positive solution of Equ. (1) converges.*

Proof. We only need to notice that since $g(u)$ is strictly concave the continuous function $G(x) := (g(u)-h(\alpha))/(u-\alpha)$ is decreasing on (α,∞). The result follows from Theorem 3.

Example 1. Consider the difference equation

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}}{1+h(x_n)} \tag{9}$$

where h is a real increasing and strictly concave continuous function defined on the interval $[0,\infty)$ such that $h(x) > 0$ for $x > 0$. Thus the function $f(x) := 1+h(x)$ is strictly concave and by Corollary 2 it follows that for each $\alpha > 0$ and $\beta \in (0, 1+h(\alpha))$ all positive solutions of Equ. (9) (with positive initial values) converge.

Example 2. Consider Equ. (1) where

$$f(u_1, u_2, ..., u_k) := 1 + u_1^{\rho_1} u_2^{\rho_2} ... u_k^{\rho_k},$$

$\alpha > 0$ and the numbers $\rho_j \in [0,1),\ j = 1,2,...,k$ are such that $\rho_1 + \rho_2 + ... + \rho_k =: \rho \in (0,1)$. Here we have $g(u) = 1 + u^\rho$, which is strictly concave since $g''(u) = \rho(\rho-1)u^{\rho-2} < 0$. Hence by Corollary 2 it follows that every positive solution converges.

Example 3. Consider Equ. (1) where

$$f(u_1, u_2, ..., u_k) := 1 + \gamma_1 u_1^{\rho_1} + \gamma_2 u_2^{\rho_2} + ... + \gamma_k u_k^{\rho_k},$$

$\alpha > 0$, $\gamma_j \geq 0$, $j = 1,2,...,k$, there is an index $j_0 \in \{1,...,k\}$ such that $\gamma_{j_0} > 0$ and $\rho_j \in [0,1),\ j = 1,2,...,k$. Here we have $g(u) := 1+\gamma_1 u^{\rho_1}+\gamma_2 u^{\rho_2}+...+\gamma_k u^{\rho_k}$, which is strictly concave since $g''(u) = \sum_{j=1}^k \gamma_j \rho_j(\rho_j - 1)u^{\rho_j - 2} < 0$. Thus the conditions of Corollary 2 are satisfied and every positive solution converges.

In the sequel we shall denote by M_1 the positive number

$$M_1 := \frac{\alpha g(\alpha)}{g(\alpha)-1}.$$

Theorem 4. *Assume $\alpha > 0$, $g(\alpha) > 1$ and the function f satisfies condition (H). Assume also that g satisfies the inequality*

$$|ug(u) - vg(v)| \leq g^2(\alpha)|u-v|, \quad u,v \in [\alpha, M_1]. \tag{10}$$

Then every solution of Equ. (1) *converges to* K.

Proof. Let (x_n) be a solution. By Theorem 2 we know that for $n \geq 1$ each term belongs to the interval $(\alpha, M_0]$, where $M_0 := M_1 + \max\{x_{-k}, ..., x_0\}$. Thus (x_n) is compact in the sense of the dynamical systems and let $(y_m), (z_m)$ be two full limiting sequences (see, e.g. [5, 6, 7, 8]) such that

$$\liminf x_n = z_0 \leq z_m, \;\; y_m \leq y_0 = \limsup x_n, \tag{11}$$

for all integers m. It is clear that the domains of both these two-sided sequences belong to the interval $[\alpha, M_0]$.

Assume that $z_0 < y_0$. From the recursive formula (1) we obtain that

$$y_0 \leq \alpha + \frac{y_0}{g(z_0)}, \;\; z_0 \geq \alpha + \frac{z_0}{g(y_0)}. \tag{12}$$

From the first inequality in (12) we obtain

$$y_0 \leq \alpha + \frac{y_0}{g(z_0)} \leq \alpha + \frac{y_0}{g(\alpha)},$$

and so

$$y_0 \leq \frac{\alpha}{1 - \frac{1}{g(\alpha)}} = M_1 \leq M_0.$$

Hence all the terms of the sequences belong to the interval $[\alpha, M_1]$.

Now from inequalities (12) and since $g(x)$ is increasing we get

$$y_0 - z_0 \leq \frac{y_0 g(y_0) - z_0 g(z_0)}{g(z_0)g(y_0)} < \frac{y_0 g(y_0) - z_0 g(z_0)}{g^2(\alpha)} \leq (y_0 - z_0),$$

which is a contradiction. Thus $y_0 = z_0$.

Theorem 5. *Assume* $\alpha > 0$, $g(\alpha) > 1$ *and the function* f *satisfies condition* (H). *Assume also that* g *is differentiable and it satisfies one of the following conditions:*

$$g'(v) < \frac{1}{\alpha}(g(v) - 1)^2, \;\; v \in [\alpha, M_1], \tag{13}$$

$$\frac{g'(v)}{g^2(v)} < \frac{1}{\alpha}\Big(1 - \frac{1}{g(\alpha)}\Big)^2, \;\; v \in [\alpha, M_1], \tag{14}$$

$$\frac{g'(v)}{g(v)(g(v)-1)} < \frac{g(\alpha) - 1}{\alpha g(\alpha)}, \;\; v \in [\alpha, M_1]. \tag{15}$$

Then every solution of Equ. (1) *converges to* K.

Proof. Let (x_n) be a solution. Then, as in Theorem 4, we let $(y_m), (z_m)$ be two full limiting sequences satisfying (11) and (12), where recall that all the terms of these two-sided sequences belong to the interval $[\alpha, M_1]$.

Assume that $z_0 < y_0$ and let condition (13) holds. Define the function

$$\Phi(u,v) := \frac{u}{\alpha} - \frac{g(v)}{g(v)-1}$$

and let

$$\phi(r) := \Phi((1-r)z_0 + ry_0, (1-r)y_0 + rz_0)).$$

Because of (12) we have $\phi(0) \geq 0 \geq \phi(1)$. Thus there is some r_0 such that $\phi'(r_0) \leq 0$. This implies that at the point $(u_0, v_0) := ((1-r_0)z_0 + r_0y_0, (1-r_0)y_0 + r_0z_0)$ it holds $\Phi_u \leq \Phi_v$. The latter says that $\alpha g'(v_0) \geq (g(v_0)-1)^2$, which is a contradiction. Thus $y_0 = z_0$.

Next assume that (14) holds. Define the function

$$\Psi(u,v) := 1 - \frac{\alpha}{u} - \frac{1}{g(v)}$$

and let

$$\psi(r) := \Psi((1-r)z_0 + ry_0, (1-r)y_0 + rz_0)).$$

Because of (12) we have $\psi(0) \geq 0 \geq \psi(1)$. Hence there is some r_0 such that $\psi'(r_0) \leq 0$. This says that at the point $(u_0, v_0) := ((1-r_0)z_0 + r_0y_0, (1-r_0)y_0 + r_0z_0)$ it holds $\Psi_u \leq \Psi_v$. The latter implies that $\alpha g^2(v_0) \leq {M_1}^2 g'(v_0)$, which contradicts to (14). Thus $y_0 = z_0$.

Finally assume that (15) holds. Define the function

$$Z(u,v) := u - \alpha - \frac{u}{g(v)}$$

and let

$$\zeta(r) := Z((1-r)z_0 + ry_0, (1-r)y_0 + rz_0)).$$

Because of (12) we have $\zeta(0) \geq 0 \geq \zeta(1)$. Thus there is some r_0 such that $\zeta'(r_0) \leq 0$. This implies that at the point $(u_0, v_0) := ((1-r_0)z_0 + r_0y_0, (1-r_0)y_0 + r_0z_0)$ it holds $Z_u \leq Z_v$. The latter says that $M_1 g'(v_0) \geq g(v_0)(g(v_0)-1)$, which is a contradiction. Thus $y_0 = z_0$.

Example 4. Assume that $g(u) = tu$. Then it is not hard to see that condition (10) is satisfied for all $\alpha > 0$ and $t > 0$ such that $t\alpha \geq 2$, while conditions (13)-(15) are satisfied for $2t\alpha \geq 3 + \sqrt{5}$.

Example 5. Let $\alpha > 1$ and assume that $g(u) = u^{\rho}$, where $\rho > 0$. Then condition (13) is satisfied for all $\rho > 0$ such that $\rho + 2 < \alpha^{\rho} + \alpha^{-\rho}$. Indeed, it is enough to show that it holds

$$\alpha\rho < v^{\rho+1} - 2v + v^{1-\rho}, \tag{16}$$

for all $v \in [\alpha, M_1]$. The right part of (16) is increasing on $[\alpha, +\infty)$, so its minimum is attained at α. Thus condition (13) holds for $\alpha\rho < \alpha^{\rho+1} - 2\alpha + \alpha^{1-\rho}$, which leads to $\rho + 2 < \alpha^{\rho} + \alpha^{-\rho}$, as desired.

Condition (14) and (15) are satisfied whenever $2\alpha^{\rho} > \rho + 2 + \sqrt{\rho^2 + 4\rho}$.

Theorem 6. *Assume $\alpha > 0$, $g(\alpha) > 1$ and that the function f satisfies condition (H). Let h be the function defined by*

$$h(u) := \frac{\alpha}{1 - \frac{1}{g(u)}}. \tag{17}$$

If the composition function $h \circ h$ is concave on the interval $[\alpha, M_0]$, then any solution of Equ. (1) converges.

Proof. Let x_n be a solution. As in Theorem 4 we obtain two full limiting sequences $(y_m), (z_m)$ with all their terms in the interval $[\alpha, M_1]$ satisfying (11) and (12). Then we have $y_0 \leq M_1$ and so, from (11) we obtain

$$z_0 \geq \alpha + \frac{z_0}{g(y_0)} \geq \alpha + \frac{z_0}{g(M_1)}.$$

Thus it holds

$$z_0 \geq \frac{\alpha}{1 - \frac{1}{g(M_1)}} = N_1.$$

Again we obtain

$$y_0 \leq \alpha + \frac{y_0}{g(z_0)} \leq \alpha + \frac{y_0}{g(N_1)},$$

and so

$$y_0 \leq \frac{\alpha}{1 - \frac{1}{g(N_1)}} = M_2,$$

as well as

$$z_0 \geq \alpha + \frac{z_0}{g(y_0)} \geq \alpha + \frac{z_0}{g(M_2)},$$

and so

$$z_0 \geq \frac{\alpha}{1 - \frac{1}{g(M_2)}} = N_2.$$

Continuing in this way we get two sequences (M_n) and (N_n) in the interval $[\alpha, M_1]$ defined as follows

$$N_n = \frac{\alpha g(M_n)}{g(M_n)-1}, \qquad M_n = \frac{\alpha g(N_{n-1})}{g(N_{n-1})-1}$$

where $M_1 = \frac{\alpha g(\alpha)}{g(\alpha)-1}$.

It is easy to show that N_n is nondecreasing and M_n is nonincreasing sequence. Thus their limits M, N respectively exist and satisfy

$$N \leq z_0 \leq y_0 \leq M,$$

as well as

$$M = h(N) \text{ and } N = h(M). \tag{18}$$

Assume that $N < M$. Then we observe that it holds $N_1 = h(h(\alpha)) > \alpha$. From the concavity, the graph of $h \circ h$ intersects the first diagonal of the plane at most one point. Consequently we have $M = N$, which implies that $z_0 = y_0$.

4 The case $g(\alpha) < 1$

In this section we shall assume that $g(\alpha) < 1$. And although it is interesting to know the behavior of all solutions of Equ. (1), we provide results only for the case $k = 2m+1$ and the odd terms of the solutions do not affect the denumerator in Equ. (1). This means that we focus our attention to the case

$$f(u_1, u_2, u_3, ..., u_{2m-1}, u_{2m}, u_{2m+1}) =: F(u_1, u_3, ..., u_{2m-1}, u_{2m+1}),$$

(where recall that f satisfies assumption (H)) and so Equ. (1) takes the form

$$x_{n+1} = a + \frac{x_{n-2m-1}}{F(x_n, x_{n-2}, x_{n-4} ..., x_{n-2m})}. \tag{19}$$

Here we have

$$g(u) := F(u, u, ..., u),$$

which is increasing. Thus so is its inverse g^{-1}. Also, as we said above, assume that $g(a) < 1$.

We have the following result:

Theorem 7. *Assume $a \geq 0$, $g(a) < 1$, the function F satisfies condition (H) and that x_n, $n = -k, -k+1, ...$ is a solution of Equ. (19) such that for all $j = 0, 1, ..., m$ it holds*

$$a \leq x_{-2j-1} < g^{-1}(1),$$

and

$$x_{-2j} > g^{-1}\Big(\frac{a}{g^{-1}(1)-a}+1\Big) =: P.$$

Then the subsequence (x_{2n}) of the even terms converges to $+\infty$, while the subsequence (x_{2n+1}) of the odd terms converges to a, provided that g is not bounded.

If g is bounded and put

$$L := \lim_{u\to+\infty} g(u),$$

then there exists a certain $b \in [a, g^{-1}(1)]$ such that any point of the recursive sequence $b_{s+1} = L(b_s - a)$, with $b_0 = b$, is a limiting point of (x_{2n+1}). In particular in case $a = 0$ we conclude the following results:

If $L < 1$, then there is a subsequence of (x_{2n+1}) which converges to zero.

If $L = 1$, then any subsequence of (x_{2n+1}) has a subsequence which converges to a period $2m+2$ solution, and

If $L > 1$, then the sequence (x_{2n+1}) converges to zero.

Proof. We observe that

$$a < x_1 = a + \frac{x_{-2m-1}}{f(x_0, x_{-2}, ..., x_{-2m})} < a + \frac{g^{-1}(1)}{g(P)} = g^{-1}(1)$$

and

$$x_2 = a + \frac{x_{-2m}}{f(x_1, x_{-1}, ..., x_{-2m+1})} > a + \frac{x_{-2m}}{g(g^{-1}(1))} = a + x_{-2m} > x_{-2m} > P.$$

Also

$$a < x_3 = a + \frac{x_{-2m+1}}{f(x_2, x_0, ..., x_{-2m+2})} < a + \frac{g^{-1}(1)}{g(P)} = g^{-1}(1)$$

and

$$x_4 = a + \frac{x_{-2m+2}}{f(x_3, x_1, ..., x_{-2m+3})} > a + \frac{x_{-2m+2}}{g(g^{-1}(1))} = a + x_{-2m+2} > x_{-2m+2} > P.$$

Repeating the same procedure by induction we obtain

$$a < x_{2r+1} = a + \frac{x_{2r-2m-1}}{f(x_{2r}, x_{2r-2}, ..., x_{2r-2m})} < a + \frac{g^{-1}(1)}{g(P)} = g^{-1}(1)$$

and

$$\begin{aligned} x_{2r+2} &= a + \frac{x_{2r-2m}}{f(x_{2r+1}, x_{2r-1}, ..., x_{2r-2m+1})} \\ &> a + \frac{x_{2r-2m}}{g(g^{-1}(1))} = a + x_{2r-2m} > x_{2r-2m} > P. \end{aligned}$$

Since the sequence of the even terms satisfies

$$x_{2(r+1)} > a + x_{2(r+1)-(2m+2)}$$

by induction we obtain that

$$x_{(2m+2)r+2j} > a + x_{(2m+2)(r-1)+2j} > \ldots > ra + x_{2j},$$

where $j \in \{-m, -(m-1)..., 0\}$.

Assume that $a > 0$. Then we get $\lim x_{2n} = +\infty$.

Suppose that the function g is not bounded, which means that $L = +\infty$. Since the sequence of the odd terms is bounded, from Equ. (19) we see that it converges to a.

Suppose that g is bounded. Then $L < +\infty$ and from Equ. (19) we get

$$\lim \frac{x_{2n+1} - a}{x_{2n-2m-1}} = \frac{1}{L}. \tag{20}$$

Now consider any subsequence $(x_{2\sigma_n+1})$ of (x_{2n+1}). Since the later is bounded, there is a $b \in [a, g^{-1}(1)]$ and a subsequence $(x_{2\mu_n+1})$ of $(x_{2\sigma_n+1})$ such that $\lim x_{2\mu_n+1} = b$. From (20) we conclude that $\lim x_{2\mu_n+1-s(2m+2)} = b_s$, where $b_{s+1} = L(b_s - a)$, for all $s = 0, 1, ...$ and $b_0 := b$.

Assume that $a = 0$. Then we have $b_{s+1} = Lb_s$, for all $s = 0, 1, ...$

If $L < 1$, then obviously we get $\lim b_s = 0$ and so 0 is a limit point of the sequence (x_{2n+1}).

Assume that $L = 1$. Then we get $b_{s+1} = b_s$, for all $s = 0, 1, ...$, which means that the sequences $(x_{2\mu_n+1-s(2m+2)})$ and $(x_{2\mu_n+1-(s+1)(2m+2)})$ have the same limits for all s. Similarly we conclude that if b^j is a limit point of the sequence $(x_{2\mu_n+1+2j})$, then b^j is also a limit point of $(x_{2\mu_n+1+2j-s(2m+2)})$, for all $j = 0, 1, ..., m$ and all $s = 0, 1, ...$. This implies that the sequence $(x_{2\mu_n+1})$ has a subsequence which converges to a $2m + 2$-cycle.

Finally, if $L > 1$, then we get $\lim b_s = +\infty$, which, in case $b > 0$, is impossible, since the sequence of the odd terms is bounded. Thus $b = 0$ for all subsequences of (x_{2n+1}). This completes the proof.

We close this section by stating the following:

Open problem 1. *Investigate the boundedness character of the nonnegative solutions of the difference equation*

$$x_{n+1} = \alpha + \frac{\beta x_{n-2k}}{g(x_n)}, \quad n = 0, 1, ...$$

where $k \in \mathbf{N}, \alpha > 0,\ \beta > g(\alpha)$ *and* $g : (0, \infty) \to (0, \infty)$ *is a real continuous increasing function on the interval* $(0, \infty)$.

5 The case $g(\alpha) = 1$

In this section we consider Equ. (1) when $g(\alpha) = 1$. If $\alpha > 0$, then as in Theorem 2 we can see that Equ. (1) has a unique positive equilibrium K in (α, ∞).

So we restrict ourselves to the case $\alpha = 0$ and $g(0) = 1$, where Equ. (1) becomes

$$x_{n+1} = \frac{x_{n-k}}{f(x_n, ..., x_{n-k+1})}, \qquad n = 0, 1, ... \tag{21}$$

In [3] the following problem is posed:

Is there a solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}, \quad x_{-1}, x_0 > 0, \quad n = 0, 1, 2, ...$$

such that $x_n \to 0$ *as* $n \to \infty$?

The positive answer to a more general problem is given in [18]. For readers who are interested in this area we leave the following problem:

Open problem 2. *Let* $g(0) = 1$ *and* $k \geq 2$. *Is there a solution of Equ.* (21) *such that* $x_n \to 0$ *as* $n \to \infty$?

We have the following result:

Theorem 8. *Assume that* $\alpha = 0$ *and* f *is continuous and increasing in all its variables. If it holds* $g(0) = 1$, *then every positive solution of Equ.* (21) *is bounded and it has a subsequence which converges to a period* $k + 1$ *solution of the form* $p, 0, 0, ..., 0, p, 0, 0, ...0, p, ...$.

Proof. Let (x_n) be a solution of Equ. (21) with positive initial values. From (21) we get

$$x_{n+1} = \frac{x_{n-k}}{f(x_n, ..., x_{n-k+1})} < \frac{x_{n-k}}{g(0)} = x_{n-k},$$

which implies that

$$x_n \leq \max\{x_{-k}, x_{-k+1}, ..., x_0\}, \quad n = -k, -(k-1), ..., 0, 1, ...$$

Therefore the solution (x_n) is bounded.

Next let (w_m) be a limiting sequence of the solution (x_n). From the general theory of discrete dynamical systems (see, [5]) we know that (w_m) also is a solution of Equ. (21). Let $(y_m), (z_m)$ be two full limiting functions of (w_n) satisfying relations (11) for (w_n). If $y_0 = 0$, then $z_0 = 0$ and so the conclusion holds for this solution, since in this case the solution (w_m) converges to 0. Assume that $y_0 > 0$. Then from (21) we obtain

$$y_0 = \frac{y_{-k-1}}{f(y_{-1}, ..., y_{-k})} \leq \frac{y_0}{f(z_0, ..., z_0)},$$

which is true only if $y_{-j} = z_0 = 0$, for all $j = 1, 2, ..., k$. Again, from (21) we obtain for all $r = 0, 1, ...$ it holds $y_{j+r(k+1)} = 0$, for $j = 1, 2, ..., k$ and $y_{r(k+1)} = y_0$. This proves the result, since any full limiting sequence of a limiting sequence is also a full limiting sequence, see [5].

Now assume that $\alpha > 0$. We have the following:

Theorem 9. *Assume that the continuous function $f(z_1, ..., z_k)$ is nondecreasing in each variable and it satisfies $g(\alpha) = 1$. Then for every nonoscillatory solution (x_n) of Equ.* (1) *the subsequences $(x_{(k+1)n+r})$, $r = 0, 1, ..., k$, are convergent.*

Proof. Let $x_n \geq K$ for all $n \in \mathbf{N} \cup \{0\}$. From (1) and by the conditions of the theorem we obtain

$$x_{n+1} - K = \frac{x_{n-k} - K}{f(x_n, ..., x_{n-k+1})} - \frac{K(f(x_n, ..., x_{n-k+1}) - g(K))}{g(K)f(x_n, ..., x_{n-k+1})}$$

$$\leq \frac{x_{n-k} - K}{f(x_n, ..., x_{n-k+1})} \leq x_{n-k} - K.$$

By this inequality we obtain that there are finite $\lim_{n\to\infty} x_{(k+1)n+r}$, $r = 0, 1, ..., k$, from which the result follows in this case. The case $x_n \leq K$, for all $n \in \mathbf{N} \cup \{0\}$, is similar and is omitted.

We finish this paper with another open problem:

Open problem 3. *Provide sufficient conditions such that when $g(\alpha) = 1$ the result of Theorem 8 holds for every oscillatory solution of Equ.* (1).

References

[1] A. M. Amleh, E. A. Grove, G. Ladas and D. A. Georgiou, On the recursive sequence $y_{n+1} = \alpha + \frac{y_{n-1}}{y_n}$, *J. Math. Anal. Appl.* **233**, (1999) 790-798.

[2] H. El-Metwally, E.A. Grove and G.Ladas, A global convergence result with applications to periodic solutions, *J. Math. Anal. Appl.* **245** (2000), 161-170.

[3] C. H. Gibbons, M.R.S. Kulenović and G.Ladas, On the recursive sequence $x_{n+1} = \frac{\alpha+\beta x_{n-1}}{\gamma+x_n}$, *Math. Sci. Res. Hot-Line* **4** (2), (2000), 1-11.

[4] C. H. Gibbons, M. R. S. Kulenović, G.Ladas and H.D.Voulov, On the trichotomy charachter of $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+x_n}$, *J. Differ. Equations Appl.* **8** (1) (2002), 75-92.

[5] G. L. Karakostas, A discrete semi-flow in the space of sequences and study of convergence of sequences defined by difference equations, *M. E. Greek Math. Soc.* **30** (1989), 66-74.

[6] G. L. Karakostas, Convergence of a difference equation via the full limiting sequences method, *Differ. Equ. Dyn. Syst.*, **1**, No. 4 (1993), pp. 289-294.

[7] G. L. Karakostas, Asymptotic 2-periodic difference equations with diagonally self-invertible responces, *J. Differ. Equations Appl.* **6** (2000), 329-335.

[8] G. L. Karakostas, On the asymptotic behavior of the difference equation $x_{n+1} = x_n\big(a - F(x_{n-1})\big)F(x_{n-2})$, *Math. Sci. Res. Hot-Line* Vol. **12** (2002), 558-572.

[9] V. L. Kocic and G. Ladas, *Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications,* Kluwer Academic Publishers, Dordrecht, 1993.

[10] M. R. S. Kulenović, G.Ladas and W. Sizer, On the recursive sequence $y_{n+1} = \frac{\alpha y_n+\beta y_{n-1}}{\gamma y_n+C y_{n-1}}$, *Math. Sci. Res. Hot-Line*, **2** (5) (1998), 1-16.

[11] G. Ladas, Open problems and conjectures, *J. Differ. Equations Appl.* **5** (1999), 211-215.

[12] S. Stević, Behaviour of the positive solutions of the generalized Beddington-Holt equation, *Panamer. Math. J.* **10** no. 4 (2000), 77-85.

[13] S. Stević, A generalization of the Copson's theorem concerning sequences which satisfy a linear inequality, *Indian J. Math.* **43** (3) (2001), 277-282.

[14] S. Stević, On the recursive sequence $x_{n+1} = -\frac{1}{x_n} + \frac{A}{x_{n-1}}$, *Int. J. Math. Math. Sci.* **27** (1) (2001), 1-6.

[15] S. Stević, A global convergence results with applications to periodic solutions, *Indian J. Pure Appl. Math.* **33** (1) (2002), 45-53.

[16] S. Stević, Asymptotic behaviour of a sequence defined by iteration with applications, *Colloq. Math.* **93** (2) (2002), 267-276.

[17] S. Stević, On the recursive sequence $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$, *Appl. Math. Lett.* **15** (2002), 305-308.

[18] S. Stević, On the recursive sequence $x_{n+1} = x_{n-1}/g(x_n)$, *Taiwanese J. Math.* **6** (3) (2002), 405-414.

[19] S. Stević, On the recursive sequence $x_{n+1} = \frac{A}{\prod_{i=0}^{k} x_{n-i}} + \frac{1}{\prod_{j=k+2}^{2(k+1)} x_{n-j}}$, *Taiwanese J. Math.* **7** (2) (2003), (to appear).

[20] S.Stević, Periodic character of a class of difference equation, (to appear).